A Class of Soliton Solutions for the N = 2 Super mKdV/Sinh-Gordon Hierarchy

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Abstract

Employing the Hirota's method, a class of soliton solutions for the $N = 2$ super mKdV equations is proposed in terms of a single Grassmann parameter. Such solutions are shown to satisfy two copies of $N = 1$ supersymmetric mKdV equations connected by nontrivial algebraic identities. Using the super Miura transformation, we obtain solutions of the $N = 2$ super KdV equations. These are shown to generalize solutions derived previously. By using the mKdV/sinh-Gordon hierarchy properties we generate the solutions of the $N = 2$ super sinh-Gordon as well.

The supersymmetric $N = 2$ Sinh-Gordon model was first introduced in [1] and [2]. Moreover, in [1] the supersymmetric $N = 2$ mKdV and its Miura transformation to the supersymmetric $N = 2$ KdV was also discussed.

In an algebraic approach, integrable hierarchies are defined by decomposition of an affine Lie algebra $\mathcal{G}$ into graded subspaces by a grading operator Q and further specified by a constant grade one element E. Such graded structure provide a systematic way to obtain solutions of the zero curvature equation for a corresponding Lax operator. For each grade one finds a solution, which corresponds to a different time evolution $t = t_k$ and hence to a different nonlinear evolution equation. In particular, supersymmetric integrable hierarchies require the decomposition of a twisted affine superalgebra. In refs. [3] and [4], the half integer decomposition of affine $\hat{sl}(2,2)$ was discussed and the equations of motion for the $N = 2$ super sinh-Gordon and mKdV were derived and shown to correspond to different time evolutions of the same hierarchy. The algebraic structure behind the hierarchy ensures universality among the solutions of different equations of motion. In fact, apart from changes of field variables, the space-time dependence of the $(2n + 1)$-th member of the mKdV/sinh-Gordon hierarchy is given by

$$\rho^{\pm}(x, t_{2n+1}) = \exp\left(\pm(2\gamma x + 2\gamma^{2n+1} t_{2n+1})\right) \tag{1}$$

and the soliton solutions of different equations of motion within the same hierarchy differ only by its space-time form specified by (1) while maintaining similar functional form.

In [5] a class of soliton solutions for the supersymmetric $N = 2$ KdV with one Grassmannian parameter was obtained employing Hirota's method. In this paper, we extend the construction of soliton solutions to the supersymmetric $N = 2$ mKdV model. The advantage of the method is that it also yields solutions to $N = 2$ super sinh-Gordon model as the space-time dependence of solutions is provided by universality of solutions ensured by the fact that both models are embedded within the same hierarchy.

By employing super Miura transformation we arrive at a more general class of $N = 2$ super KdV equation which for a particular choice of parameters agrees with the one obtained in [5].

The $N = 2$ super mKdV model is described by the $t = t_3$ flow of the affine $\hat{sl}(2,2)$ hierarchy (see [3]) :

$$\begin{aligned} 4\partial_{t_3}\psi_1 &= \partial_x^3\psi_1 - 3(u_1^2+u_3^2)\partial_x\psi_1 - \frac{3}{2}\partial_x(u_1^2+u_3^2)\psi_1 - 3\partial_x(u_1u_3)\psi_3 \\ 4\partial_{t_3}u_1 &= \partial_x^3 u_1 + \partial_x\left[-2u_1^3 + 3u_1(\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3) - 3u_3\partial_x(\psi_1\psi_3)\right] \\ 4\partial_{t_3}\psi_3 &= \partial_x^3\psi_3 - 3(u_1^2+u_3^2)\partial_x\psi_3 - \frac{3}{2}\partial_x(u_1^2+u_3^2)\psi_3 - 3\partial_x(u_1u_3)\psi_1 \\ 4\partial_{t_3}u_3 &= \partial_x^3 u_3 + \partial_x\left[-2u_3^3 - 3u_3(\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3) + 3u_1\partial_x(\psi_1\psi_3)\right]\,. \end{aligned} \tag{2}$$

The $N = 2$ super sinh-Gordon model belongs to the same hierarchy but with the flow parameter $t = t_{-1}$ for which one finds the following evolution equations :

$$\begin{aligned} \partial_{t_{-1}}\partial_x(\phi_1 \pm \phi_3) &= 4\sinh(\phi_1 \pm \phi_3)\cosh(\phi_1 \mp \phi_3) - 4(\psi_1 \pm \psi_3)(\bar\psi_1 \pm \bar\psi_3)\sinh(\phi_1 \mp \phi_3), \\ \partial_{t_{-1}}(\psi_1 \pm \psi_3) &= -2(\bar\psi_1 \mp \bar\psi_3)\cosh(\phi_1 \pm \phi_3)\,, \end{aligned} \tag{3}$$

where $\bar\psi_{1,3}$ are auxiliary fields satisfying

$$\partial_x(\bar\psi_1 \pm \bar\psi_3) = -2(\psi_1 \mp \psi_3)\cosh(\phi_1 \pm \phi_3)\,. \tag{4}$$

The fact that both integrable models belong to the same hierarchy is expressed by relation

$$u_i = -\partial_x\phi_i, \quad i = 1,3\,. \tag{5}$$

Define now the superfields

$$\chi_1 = \psi_1 + \theta u_1, \qquad \chi_3 = \psi_3 - \theta u_3, \tag{6}$$

and the superderivative

$$D = \partial_\theta + \theta\partial_x, \qquad D^2 = \partial_x, \tag{7}$$

The $N = 2$ supersymmetric mKdV equations can be recast as

$$\begin{aligned} 4\partial_{t_3}\chi_1 &= \partial_x^3\chi_1 + D\left[-2(D\chi_1)^3 + 3(\chi_1\partial_x\chi_1 - \chi_3\partial_x\chi_3)D\chi_1 + 3D\chi_3\partial_x(\chi_1\chi_3)\right] \\ 4\partial_{t_3}\chi_3 &= \partial_x^3\chi_3 + D\left[-2(D\chi_3)^3 - 3(\chi_1\partial_x\chi_1 - \chi_3\partial_x\chi_3)D\chi_3 - 3D\chi_1\partial_x(\chi_1\chi_3)\right] \end{aligned} \tag{8}$$

We now introduce the following tau functions :

$$\chi_1 = D\ln\left(\frac{\tau_1}{\tau_2}\right), \qquad \chi_3 = D\ln\left(\frac{\tau_3}{\tau_4}\right) \tag{9}$$

and the Hirota's derivatives

$$\begin{aligned}
\mathbf{SD}_{t_3}(\tau_1.\tau_1) &= 2(D\partial_{t_3}\tau_1\,\tau_1 - D\tau_1\partial_{t_3}\tau_1),\\
\mathbf{SD}_{x}(\tau_1.\tau_1) &= 2(D\partial_{x}\tau_1\,\tau_1 - D\tau_1\partial_{x}\tau_1),\\
\mathbf{SD}^3_{x}(\tau_1.\tau_1) &= 2(D\partial^3_{x}\tau_1\,\tau_1 - 3D\partial^2_x\tau_1\partial_x\tau_1 + 3D\partial_x\tau_1\partial^2_x\tau_1 - D\tau_1\partial^3_x\tau_1),\\
\mathbf{D}^2_x(\tau_1.\tau_2) &= \partial^2_x\tau_1\,\tau_2 - 2\partial_x\tau_1\partial_x\tau_2 + \tau_1\partial^2_x\tau_2,\\
\mathbf{D}^2_x(\tau_1.\tau_1) &= 2\left[\partial^2_x\tau_1\,\tau_1 - (\partial_x\tau_1)^2\right],\\
\bar{\mathbf{D}}(\tau_a.\tau_b) &= D\tau_a\partial_x\tau_b - D\tau_b\partial_x\tau_a, \quad a \neq b = 1,2,3,4\,.
\end{aligned} \tag{10}$$

The first of equations in (8) becomes

$$\begin{aligned}
&2\left[\frac{\mathbf{SD}_{t_3}(\tau_1.\tau_1)}{\tau_1^2} - \frac{\mathbf{SD}_{t_3}(\tau_2.\tau_2)}{\tau_2^2}\right] = \frac{\mathbf{SD}^3_x(\tau_1.\tau_1)}{2\tau_1^2} - \frac{\mathbf{SD}^3_x(\tau_2.\tau_2)}{2\tau_2^2}\\
&-\frac{3}{2}\left[\frac{\mathbf{D}^2_x(\tau_1.\tau_2)}{\tau_1\tau_2} + \frac{\mathbf{D}^2_x(\tau_3.\tau_4)}{\tau_3\tau_4}\right]\left[\frac{\mathbf{SD}_x(\tau_1.\tau_1)}{\tau_1^2} - \frac{\mathbf{SD}_x(\tau_2.\tau_2)}{\tau_2^2}\right]\\
&-\frac{3}{2}\left[\frac{\mathbf{D}^2_x(\tau_1.\tau_1)}{\tau_1^2} - \frac{\mathbf{D}^2_x(\tau_2.\tau_2)}{\tau_2^2}\right]\left[\frac{\mathbf{SD}_x(\tau_1.\tau_2)}{\tau_1\tau_2} - \frac{\mathbf{SD}_x(\tau_3.\tau_4)}{\tau_3\tau_4}\right]\\
&+\frac{3}{4}\left[\frac{\mathbf{D}^2_x(\tau_3.\tau_3)}{\tau_3^2}\frac{\mathbf{SD}_x(\tau_1.\tau_1)}{\tau_1^2} - \frac{\mathbf{D}^2_x(\tau_1.\tau_1)}{\tau_1^2}\frac{\mathbf{SD}_x(\tau_3.\tau_3)}{\tau_3^2}\right]\\
&-\frac{3}{4}\left[\frac{\mathbf{D}^2_x(\tau_3.\tau_3)}{\tau_3^2}\frac{\mathbf{SD}_x(\tau_2.\tau_2)}{\tau_2^2} - \frac{\mathbf{D}^2_x(\tau_2.\tau_2)}{\tau_2^2}\frac{\mathbf{SD}_x(\tau_3.\tau_3)}{\tau_3^2}\right]\\
&+\frac{3}{4}\left[\frac{\mathbf{D}^2_x(\tau_4.\tau_4)}{\tau_4^2}\frac{\mathbf{SD}_x(\tau_1.\tau_1)}{\tau_1^2} - \frac{\mathbf{D}^2_x(\tau_1.\tau_1)}{\tau_1^2}\frac{\mathbf{SD}_x(\tau_4.\tau_4)}{\tau_4^2}\right]\\
&-\frac{3}{4}\left[\frac{\mathbf{D}^2_x(\tau_4.\tau_4)}{\tau_4^2}\frac{\mathbf{SD}_x(\tau_2.\tau_2)}{\tau_2^2} - \frac{\mathbf{D}^2_x(\tau_2.\tau_2)}{\tau_2^2}\frac{\mathbf{SD}_x(\tau_4.\tau_4)}{\tau_4^2}\right]\\
&+\frac{3}{2}\left(\frac{\mathbf{D}^2_x(\tau_3.\tau_3)}{\tau_3^2} - \frac{\mathbf{D}^2_x(\tau_4.\tau_4)}{\tau_4^2}\right)\left(\frac{\bar{\mathbf{D}}(\tau_3.\tau_1)}{\tau_3\tau_1} - \frac{\bar{\mathbf{D}}(\tau_3.\tau_2)}{\tau_3\tau_2} - \frac{\bar{\mathbf{D}}(\tau_4.\tau_1)}{\tau_4\tau_1} + \frac{\bar{\mathbf{D}}(\tau_4.\tau_2)}{\tau_4\tau_2}\right).
\end{aligned} \tag{11}$$

The second of equations in (8) is obtained through the transformation $\tau_1 \leftrightarrow \tau_3$ and $\tau_2 \leftrightarrow \tau_4$.

We will discuss a class of solutions of eqn. (11) satisfying

$$\begin{aligned}
(4\mathbf{SD}_{t_3} - \mathbf{SD}^3_x)(\tau_a.\tau_a) &= 0, \qquad \text{for} \quad a = 1,2,3,4\\
\mathbf{D}^2_x(\tau_1.\tau_2) &= 0\\
\mathbf{D}^2_x(\tau_3.\tau_4) &= 0\\
\mathbf{SD}_x(\tau_1.\tau_2) &= 0\\
\mathbf{SD}_x(\tau_3.\tau_4) &= 0\\
\mathbf{D}^2_x(\tau_a.\tau_a)\mathbf{SD}_x(\tau_b.\tau_b) - \mathbf{D}^2_x(\tau_b.\tau_b)\mathbf{SD}_x(\tau_a.\tau_a) &= 0, \qquad \text{for} \quad a = 3,4 \quad b = 1,2\\
\bar{\mathbf{D}}(\tau_a.\tau_b) &= 0, \qquad \text{for} \quad a = 3,4 \quad b = 1,2
\end{aligned} \tag{12}$$

Let all τ_i, $i = 1, ..., 4$ be of the form $1 + \Sigma$ where Σ is a combination of exponential functions of $\tilde{\eta}_a = 2k_a x + w_a t + \zeta_a \theta$, $a = 1, ..., 4$ with constant parameters k_a, w_a and Grassmann parameters ζ_a. In order to illustrate the method below we consider two explicit examples.

- *Two parameter solution*

  Consider the following ansatz

$$\begin{aligned} \tau_1 &= 1 + \alpha_1 e^{\tilde{\eta}_1}, \qquad \tau_2 = 1 + \alpha_2 e^{\tilde{\eta}_2}, \\ \tau_3 &= 1 + \alpha_3 e^{\tilde{\eta}_3}, \qquad \tau_4 = 1 + \alpha_4 e^{\tilde{\eta}_4}\,. \end{aligned} \tag{13}$$

  Using the relations,

$$\begin{aligned} \mathbf{SD}_x^n(e^{\tilde{\eta}_1}.e^{\tilde{\eta}_2}) &= (2k_1 - 2k_2)^n[-(\zeta_1 - \zeta_2) + 2\theta(k_1 - k_2)]e^{\tilde{\eta}_1+\tilde{\eta}_2}, \\ \mathbf{D}_x^n(e^{\tilde{\eta}_1}.e^{\tilde{\eta}_2}) &= (2k_1 - 2k_2)^n e^{\tilde{\eta}_1+\tilde{\eta}_2}, \\ \mathbf{SD}_{t_3}^n(e^{\tilde{\eta}_1}.e^{\tilde{\eta}_2}) &= (\omega_1 - \omega_2)^n[-(\zeta_1 - \zeta_2) + 2\theta(k_1 - k_2)]e^{\tilde{\eta}_1+\tilde{\eta}_2}, \\ \bar{\mathbf{D}}(e^{\tilde{\eta}_1}.e^{\tilde{\eta}_2}) &= 2(-\zeta_2 k_1 + \zeta_1 k_2)e^{\tilde{\eta}_1+\tilde{\eta}_2}, \end{aligned} \tag{14}$$

  we verify that eqns. (12) are satisfied if

$$\begin{aligned} & k_2 = k_1, && k_4 = k_3, \\ & \zeta_2 = \zeta_1, && \zeta_4 = \zeta_3, \\ & \omega_1 = \omega_2 = 2k_1^3, && \omega_3 = \omega_4 = 2k_3^3, \\ & \alpha_2 = -\alpha_1, && \alpha_4 = -\alpha_3, \\ & \zeta_3 = \frac{k_3}{k_1}\zeta_1. \end{aligned} \tag{15}$$

  Explicitly, we find

$$\begin{aligned} u_1 &= 2k_1\alpha_1 e^{\eta_1}\left(\frac{1}{1+\alpha_1 e^{\eta_1}} + \frac{1}{1-\alpha_1 e^{\eta_1}}\right), \\ u_3 &= -2k_3\alpha_3 e^{\eta_3}\left(\frac{1}{1+\alpha_3 e^{\eta_3}} + \frac{1}{1-\alpha_3 e^{\eta_3}}\right), \\ \psi_1 &= -\zeta_1\alpha_1 e^{\eta_1}\left(\frac{1}{1+\alpha_1 e^{\eta_1}} + \frac{1}{1-\alpha_1 e^{\eta_1}}\right), \\ \psi_3 &= -\zeta_1\frac{k_3}{k_1}\alpha_3 e^{\eta_3}\left(\frac{1}{1+\alpha_3 e^{\eta_3}} + \frac{1}{1-\alpha_3 e^{\eta_3}}\right), \end{aligned} \tag{16}$$

  where ζ_1 denotes the single Grassmann parameter and

$$\eta_a = 2(k_a x + k_a^3 t_3), \qquad a = 1, 3. \tag{17}$$

- *Four parameter solution*

Consider the following ansatz

$$
\begin{aligned}
\tau_1 &= 1 + \alpha_1 e^{\tilde{\eta}_1} + \alpha_2 e^{\tilde{\eta}_2} + \alpha_1\alpha_2 A_{1,2} e^{\tilde{\eta}_1+\tilde{\eta}_2}, \\
\tau_2 &= 1 + \beta_1 e^{\tilde{\eta}_1} + \beta_2 e^{\tilde{\eta}_2} + \beta_1\beta_2 B_{1,2} e^{\tilde{\eta}_1+\tilde{\eta}_2}, \\
\tau_3 &= 1 + \alpha_3 e^{\tilde{\eta}_3} + \alpha_4 e^{\tilde{\eta}_4} + \alpha_3\alpha_4 A_{3,4} e^{\tilde{\eta}_3+\tilde{\eta}_4}, \\
\tau_4 &= 1 + \beta_3 e^{\tilde{\eta}_3} + \beta_4 e^{\tilde{\eta}_4} + \beta_3\beta_4 B_{3,4} e^{\tilde{\eta}_3+\tilde{\eta}_4}\,.
\end{aligned} \tag{18}
$$

Substituting in eqn. (12), we find

$$
\beta_s = -\alpha_s, \qquad \omega_s = 2k_s^3, \quad s = 1,2,3,4 \tag{19}
$$

$$
B_{i,j} = A_{i,j} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \qquad k_l\zeta_m = k_m\zeta_l, \tag{20}
$$

for $(i=1, j=2)$, $(i=3, j=4)$ and $l,m = 1,2,3,4$. Conditions (15) and (20) justify the presence of a single Grassmann parameter ζ_1. In components we have

$$
\tau_k = \tau_k^a + \tau_k^b\zeta_1\theta, \qquad k = 1,2,3,4 \tag{21}
$$

for which we obtain explicitly,

$$
\begin{aligned}
\tau_1^a &= 1 + \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_1\alpha_2 A_{1,2} e^{\eta_1+\eta_2}, \\
\tau_2^a &= 1 - \alpha_1 e^{\eta_1} - \alpha_2 e^{\eta_2} + \alpha_1\alpha_2 A_{1,2} e^{\eta_1+\eta_2}, \\
\tau_3^a &= 1 + \alpha_3 e^{\eta_3} + \alpha_4 e^{\eta_4} + \alpha_3\alpha_4 A_{3,4} e^{\eta_3+\eta_4}, \\
\tau_4^a &= 1 - \alpha_3 e^{\eta_3} - \alpha_4 e^{\eta_4} + \alpha_3\alpha_4 A_{3,4} e^{\eta_3+\eta_4}, \\
\tau_1^b &= \frac{1}{k_1}\left(\alpha_1 k_1 e^{\eta_1} + \alpha_2 k_2 e^{\eta_2} + \alpha_1\alpha_2 (k_1+k_2) A_{1,2} e^{\eta_1+\eta_2}\right), \\
\tau_2^b &= \frac{1}{k_1}\left(-\alpha_1 k_1 e^{\eta_1} - \alpha_2 k_2 e^{\eta_2} + \alpha_1\alpha_2 (k_1+k_2) A_{1,2} e^{\eta_1+\eta_2}\right), \\
\tau_3^b &= \frac{1}{k_1}\left(\alpha_3 k_3 e^{\eta_3} + \alpha_4 k_4 e^{\eta_4} + \alpha_3\alpha_4 (k_3+k_4) A_{3,4} e^{\eta_3+\eta_4}\right), \\
\tau_4^b &= \frac{1}{k_1}\left(-\alpha_3 k_3 e^{\eta_3} - \alpha_4 k_4 e^{\eta_4} + \alpha_3\alpha_4 (k_3+k_4) A_{3,4} e^{\eta_3+\eta_4}\right),
\end{aligned} \tag{22}
$$

where ζ_1 is a constant fermionic parameter and

$$
\eta_a = 2(k_a x + k_a^3 t_3). \tag{23}
$$

From (6) and (9) we find

$$
\begin{aligned}
u_1 &= \partial_x \ln\left(\frac{\tau_1^a}{\tau_2^a}\right), & u_3 &= -\partial_x \ln\left(\frac{\tau_3^a}{\tau_4^a}\right), \\
\psi_1 &= \zeta_1\left(\frac{\tau_2^b}{\tau_2^a} - \frac{\tau_1^b}{\tau_1^a}\right), & \psi_3 &= \zeta_1\left(\frac{\tau_4^b}{\tau_4^a} - \frac{\tau_3^b}{\tau_3^a}\right),
\end{aligned} \tag{24}
$$

Equation (24) together with equations (22)-(23) provide a class of solutions for fields (u_1, ψ_1) and (u_3, ψ_3), which satisfy both $N = 1$ and $N = 2$ super mKdV equations of motion since they also satisfy the non trivial relations like

$$
\begin{aligned}
u_3^2 \partial_x \psi_1 + \frac{1}{2} \partial_x (u_3^2) \psi_1 + \partial_x (u_1 u_3) \psi_3 &= 0, \\
u_1^2 \partial_x \psi_3 + \frac{1}{2} \partial_x (u_1^2) \psi_3 + \partial_x (u_1 u_3) \psi_1 &= 0 \,.
\end{aligned}
\tag{25}
$$

In general, identities (25) follow directly from conditions (12).

We now discuss the corresponding soliton solutions for the $N = 2$ super sinh-Gordon (3) and (4). Following the arguments of ref. [3], where it was shown that the mKdV and sinh-Gordon models belong to the same integrable hierarchy, and taking into account the space-time dependence given in equation (1) we relate solutions of both models to each other by replacing in (23)

$$
k_a^3 t_3 \to k_a{}^{-1} t_{-1}, \quad i.e., \quad \eta_a = 2(k_a x + k_a^{-1} t_{-1}) \tag{26}
$$

and $u_i = -\partial_x \phi_1, \;\; i = 1, 3$. Henceforth

$$
\phi_1 = -\ln \left(\frac{\tau_1^a}{\tau_2^a} \right), \qquad \phi_3 = \ln \left(\frac{\tau_3^a}{\tau_4^a} \right). \tag{27}
$$

Plugging (22) in (27) and taking into account (26) we obtain fields ϕ_1, ϕ_3. The fermionic fields ψ_1 and ψ_3 are obtained from (24) with the space-time dependence given by (26). The auxiliary fields $\bar{\psi}_1$ and $\bar{\psi}_3$ are then solved by the second of eqns. (3) yielding,

$$
\begin{aligned}
\bar{\psi}_1 &= \frac{1}{2} \left[\frac{(\partial_{t_{-1}} \psi_3) \mathrm{sh}\phi_1 \, \mathrm{sh}\phi_3 - (\partial_{t_{-1}} \psi_1) \mathrm{ch}\phi_1 \, \mathrm{ch}\phi_3}{(\mathrm{ch}\phi_1 \, \mathrm{ch}\phi_3)^2 - (\mathrm{sh}\phi_1 \, \mathrm{sh}\phi_3)^2} \right], \\
\bar{\psi}_3 &= \frac{1}{2} \left[\frac{(\partial_{t_{-1}} \psi_3) \mathrm{ch}\phi_1 \, \mathrm{ch}\phi_3 - (\partial_{t_{-1}} \psi_1) \mathrm{sh}\phi_1 \, \mathrm{sh}\phi_3}{(\mathrm{ch}\phi_1 \, \mathrm{ch}\phi_3)^2 - (\mathrm{sh}\phi_1 \, \mathrm{sh}\phi_3)^2} \right].
\end{aligned}
\tag{28}
$$

It is interesting to consider as a particular example, the case of $\alpha_1 = \alpha_3 = 0$ for which we obtain,

$$
\begin{aligned}
\bar{\psi}_1 &= \frac{2 \zeta_1 e^{\eta_2} \, \alpha_2 \, (1 + e^{2\,\eta_4} \, {\alpha_4}^2)}{k_1 \, (-1 + e^{2\,\eta_2} \, {\alpha_2}^2) \, (-1 + e^{2\,\eta_4} \, {\alpha_4}^2)}, \\
\bar{\psi}_3 &= -\frac{2 \zeta_1 e^{\eta_4} \, (1 + e^{2\,\eta_2} \, {\alpha_2}^2) \, \alpha_4}{k_1 \, (-1 + e^{2\,\eta_2} \, {\alpha_2}^2) \, (-1 + e^{2\,\eta_4} \, {\alpha_4}^2)},
\end{aligned}
\tag{29}
$$

We have explicitly verified that the formulae (22) with evolution parameter t_{-1} given by (26) and general values of parameters $\alpha_i, i = 1, 2, 3, 4$ in equation (28) indeed gives the solutions to the $N = 2$ super sinh-Gordon equations (3).

We now relate the above solutions to solutions of the super $N = 2$ KdV equation. Define two spin 1/2 superfields $\Psi, i = 1, 2$ as

$$
\Psi_1 = \chi_1 + \chi_3, \qquad \Psi_2 = \chi_1 - \chi_3 \,. \tag{30}
$$

Eqn. (8) gives

$$
\begin{aligned}
4\partial_{t_3}\Psi_1 &= D\left[\partial_x^2 D\Psi_1 + 3\Psi_1\partial_x\Psi_2 D\Psi_2 - \frac{1}{2}(D\Psi_1)^3 - \frac{3}{2}D\Psi_1(D\Psi_2)^2\right], \\
4\partial_{t_3}\Psi_2 &= D\left[\partial_x^2 D\Psi_2 + 3\Psi_2\partial_x\Psi_1 D\Psi_1 - \frac{1}{2}(D\Psi_2)^3 - \frac{3}{2}D\Psi_2(D\Psi_1)^2\right].
\end{aligned}
\tag{31}
$$

These equations, after time rescaling $t_3 \to -4t_3$, become eqns. (4.6) of ref. [1]. Introducing the $N = 2$ super Miura transformation given in eqn. (3.9) of [1], i.e.

$$
\begin{aligned}
U &= D(\Psi_1 + \Psi_2) - \Psi_1\Psi_2 = 2D\chi_1 + 2\chi_1\chi_3, \\
V &= \partial_x\Psi_2 - \Psi_2 D\Psi_1 = \partial_x\chi_1 - \partial_x\chi_3 - \chi_1 D\chi_1 - \chi_1 D\chi_3 + \chi_3 D\chi_1 + \chi_3 D\chi_3
\end{aligned}
\tag{32}
$$

yields the $N = 2$ super KdV equations of [1],

$$
\begin{aligned}
4\partial_{t_3}U &= \partial_x\left[\partial_x^2 U + 3(DU)V - \frac{1}{2}U^3\right], \\
4\partial_{t_3}V &= \partial_x\left[\partial_x^2 V - 3V(DV) + 3V\partial_x U - \frac{3}{2}VU^2\right]
\end{aligned}
\tag{33}
$$

for the spin 1 and 3/2 superfields, respectively. Let the U and V be decomposed as follows

$$
U = U^b + \theta U^f \qquad V = V^f + \theta V^b,
$$

with indices b and f referring to boson and fermion components, respectively.

Using (6) in the r.h.s. of (32) we obtain

$$
\begin{aligned}
U^b &= 2(u_1 + \psi_1\psi_3) \\
U^f &= 2(\partial_x\psi_1 + u_3\psi_1 + u_1\psi_3)
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
V^f &= \partial_x(\psi_1 - \psi_3) - (u_1 - u_3)(\psi_1 - \psi_3) \\
V^b &= \partial_x(u_1 + u_3) - (u_1^2 - u_3^2) + (\psi_1 - \psi_3)\partial_x(\psi_1 + \psi_3).
\end{aligned}
\tag{35}
$$

As an example we set $\alpha_2 = \alpha_4 = 0$ in eqs. (22). Then equations (34) and (35) give

$$
\begin{aligned}
U^b &= \frac{8\,e^{\eta_1}\,k_1\,\alpha_1}{1 - e^{2\,\eta_1}\,{\alpha_1}^2}, \\
U^f &= \frac{-8\,\zeta_1\,e^{\eta_1}\,k_1\,\alpha_1\,\left(1 + e^{2\,\eta_1}\,{\alpha_1}^2\right)}{\left(-1 + e^{2\,\eta_1}\,{\alpha_1}^2\right)^2}, \\
V^b &= \frac{-8\,\left(e^{\eta_3}\,{k_3}^2\,(1 + e^{\eta_1}\,\alpha_1)^2\,\alpha_3 - e^{\eta_1}\,{k_1}^2\,\alpha_1\,(1 + e^{\eta_3}\,\alpha_3)^2\right)}{\left(1 + e^{\eta_1}\,\alpha_1\right)^2\,\left(1 + e^{\eta_3}\,\alpha_3\right)^2}, \\
V^f &= \frac{4\,\zeta_1\,\left(e^{\eta_3}\,{k_3}^2\,(1 + e^{\eta_1}\,\alpha_1)^2\,\alpha_3 - e^{\eta_1}\,{k_1}^2\,\alpha_1\,(1 + e^{\eta_3}\,\alpha_3)^2\right)}{k_1\,\left(1 + e^{\eta_1}\,\alpha_1\right)^2\,\left(1 + e^{\eta_3}\,\alpha_3\right)^2}.
\end{aligned}
\tag{36}
$$

Since we are considering the class of solutions with only one Grassmann constant parameter, the fermionic quadratic terms in (35) vanish identically. Furthermore, for the solutions given in (16) and in (22) since they satisfy $u_3\psi_1 + u_1\psi_3 = 0$ (which can be checked in general using (12)), it follows that U^b and U^f depend only on α_1, α_2 and V^b and V^f depend upon $\alpha_1, \alpha_2, \alpha_4$ and α_4. In particular, for $\alpha_3 = \alpha_4 = 0$ (with $t_3 \to -4t_3, k \to k/2$), using algebraic computer methods, we have checked that our solutions agree with those found in [5]. The other solutions, like, for example those given in equation (36) with at least one of the parameters α_3, α_4 being different from zero are, as far as we are aware, new solutions.

It would be interesting to generalize this construction to involve multiple Grassmann constant parameters. In analogy to what was done for the corresponding $N = 1$ hierarchy in [6], this may be accomplished in terms of vertex operators and representations of the $\hat{sl}(2,2)$ affine algebra.

Acknowledgements
We are grateful to CNPq and FAPESP for financial support. The work is also supported by grant NSF PHY-0651694.

References

[1] T. Inami and H. Kanno, *Nucl. Phys.* **B359** (1990) 201

[2] K. Kobayashi and T. Uematsu, *Phys. Lett.* **264B** (2001) 107

[3] H. Aratyn, J.F. Gomes, L. H. Ymai and A.H. Zimerman,"$N = 2$ and $N = 4$ Supersymmetric mKdV and sinh-Gordon Hierarchies", hep-th/0409171

[4] H. Aratyn, J.F. Gomes, and A.H. Zimerman, *Nucl. Phys.* **B676** (2004) 537

[5] S. Ghosh and D. Sarma, *Phys. Lett.* **522B** (2001) 189 ,

[6] J.F. Gomes, L. H. Ymai and A.H. Zimerman, *Phys. Lett.* **359A** (2006) 630, hep-th/0607107